PR-003-1164001 Seat No. ________

M. Sc. (Sem. IV) (CBCS) Examination

August - 2020

Mathematics : CMT - 4001

(Linear Algebra)

Faculty Code : 003
Subject Code : 1164001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Attempt all the questions.
(2) There are 5 questions.
(3) Figures to the right indicate full marks.

1 Answer any **seven** : **14**

(1) If a linear transformation T is left invertible then show that T is invertible.

(2) Let $T: P_2[x] \to P_3[x]$ be a linear transformation defined by $T(p(x)) = \int p(x)\,dx$. Find matrix of T in the standard bases.

(3) Suppose that T is a nilpotent linear transformation and $\alpha \in F$ is non-zero then prove that $\alpha I + T$ is regular.

(4) For any $A \in M_n(\mathbb{C})$ show that tr $(AA^*) \geq 0$.

(5) Prove or disprove :

(1) $tr(AB) = tr(A)tr(B)$

(2) $\det(A+B) = \det(A) + \det(B)$.

(6) Let V be an inner product space. Then show that $(S+T)^* = S^* + T^*$ and $(ST)^* = S^*T^*$.

(7) Characterise eigen values of a unitary transformation.

(8) Define : Bilinear form and non-degenerate bilinear form.

(9) Prove that any orthonormal subset of an inner product space is linearly independent.

(10) Suppose that f is a non-zero skew-symmetric bilinear form and $v, w \in V$ be such that $f(v, w) = 1$. Then show that v and w are linearly independent.

2 Attempt any **two** : **14**

(1) Prove that $p(x)$ is a minimal polynomial for T over F if and only if whenever $h(x) \in F[x]$ such that $h(T) = 0$, $p(x)$ divides $h(x)$.

(2) State and prove :

(1) Jacobson's lemma

(2) Polarization identity.

(3) Suppose T is nilpotent with index of nilpotence n_1 and let $v \in V$ be such that $T^{n_1 - 1}(v) \neq 0$. Then show that, $V_1 = L(\{v, Tv, \ldots, T^{n_1 - 1}v\})$ is a subspace of V with dimension n_1 and it is invariant under T. Also find the matrix of $T \mid v_1$.

3 Answer the following : **14**

(1) Prove that : A linear transformation T is invertible if and only if the constant term in the minimal polynomial for T is not 0.

(2) Prove that T is unitary if and only if T maps an orthonormal basis of V to an orthonormal basis of V.

OR

3 Answer the following : **14**

(1) Prove that T is regular if and only if $\ker(T) = \{0\}$.

(2) State and prove : Cramer's rule.

4 Answer the following : **14**

(1) Suppose V_1 and V_2 are invariant subspaces of V under T such that $V = V_1 \oplus V_2$. If $p_1(x), p_2(x) \in F[x]$ are minimal polynomial for $T|v_1$ and $T|v_2$ respectively. Then prove that minimal polynomial for T is the LCM of $p_1(x)$ and $p_2(x)$.

(2) Let $A, B \in M_n(F)$. Then prove that, $\det(AB) = \det(A)\det(B)$.

5 Attempt any **two** : **14**

(1) Prove that λ is characteristic root of T if and only if λ is a root of minimal polynomial for T.

(2) Suppose that V is a cyclic $F[x]$–module and $p(x) \in F[x]$ is the minimal polynomial for T. Then prove that there exists a basis of V over F such that matrix of T is companion matrix of $p(x)$.

(3) Show that any eigen value of a Hermitian matrix over $\mathbb{C}$ is real. Using this result deduce that if the matrix is of the form AA^* for some $A \in M_n(\mathbb{C})$, then its eigen value is non-negative.

(4) Let f be a bilinear form then prove that f is symmetric if and only if $[f]_B$ is symmetric for any basis B of V over F.